

Remarks on the cone of divisors

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Abstract

We prove two theorems on the locally finite decompositions of the cones of divisors by the cones which correspond to canonical and minimal models. We introduce the concept of the numerical linear systems in order to simplify the argument on the Zariski decompositions.

1 Introduction

In the Minimal Model Program (MMP), a contraction morphism arises from an extremal ray on a *cone of curves*, a subset of a vector space $N_1(X/S)$ consisting of numerical classes of effective 1-cycles. This is like a homology theory. According to the later development of the MMP, it turned out that the subsets of the dual vector space $N^1(X/S)$, the vector space of divisors, are more useful. This is like a cohomology theory. In this paper we shall make some remarks on the finiteness properties on subsets of $N^1(X/S)$.

Let $f : X \rightarrow S$ be a projective morphism between algebraic schemes. Two \mathbf{R} -Cartier divisors D_1 and D_2 , linear combinations of Cartier divisors with real coefficients, on X are said to be *numerically equivalent* over S , and denoted by $D_1 \equiv_S D_2$, or simply $D_1 \equiv D_2$, if equalities $(D_1 \cdot C) = (D_2 \cdot C)$ hold for all curves C on X which are mapped to points on S . The set of all the numerical classes of \mathbf{R} -Cartier divisors form a finite dimensional real vector space $N^1(X/S)$. We consider the following inclusions of convex cones inside $N^1(X/S)$:

$$\begin{array}{ccc} \mathrm{Amp}(X/S) & \longrightarrow & \mathrm{Nef}(X/S) \\ \downarrow & & \downarrow \\ \mathrm{Big}(X/S) & \longrightarrow & \mathrm{Psef}(X/S) \end{array}$$

where the *ample cone* $\text{Amp}(X/S)$ is the open convex cone generated by the classes of Cartier divisors which are ample over S , the *nef cone* $\text{Nef}(X/S)$ is its closure, the *pseudo-effective cone* $\text{Psef}(X/S)$ is the closed convex cone generated by the classes of effective Cartier divisors, and the *big cone* $\text{Big}(X/S)$ is its interior (cf. [4]).

We shall prove two theorems on the locally finite decompositions of these cones in §4 which are determined respectively by the canonical and minimal models (Theorems 3 and 4). The assertions of the theorems are basically contained in a paper by Shokurov [8]. But we make the statements more precise, especially in the second theorem. We considered a partial decomposition of the vector space $N^1(X/S)$ by nef cones of various birationally equivalent minimal models in [4]. That was a locally finite decomposition. Shokurov's idea is to consider the space of boundary divisors themselves instead of their numerical classes, so that the assertions of theorems become not only local finiteness but also global finiteness. We treat log pairs whose boundaries are not necessarily big, but we assume that there exist minimal and canonical models of the pairs (cf. [1]). We include examples at the end of the paper on the whole space $N^1(X/S)$ where the finiteness holds only locally.

We introduce the concept of the numerical linear systems in §3 which replaces that of the \mathbf{R} -linear systems in [1] in order to simplify the argument on the Zariski decompositions. We identify the exceptional divisors of the birational map to a minimal model as the numerical fixed part of the numerical canonical system (Lemma 2). This lemma is used to prove the finite decomposition theorem according to the choices of minimal models (Theorem 4).

2 Minimal and canonical models

The purpose of this section is to fix the notation and the terminology. The MMP deals with pairs (X, B) consisting of normal varieties and \mathbf{R} -divisors. Let $\mu : Y \rightarrow X$ be a log resolution of the pair (X, B) , i.e., μ is a birational morphism from a smooth variety Y such that the inverse image of B and the exceptional locus of μ as well as their union are normal crossing divisors. We can write $\mu^*(K_X + B) = K_Y + C$ with irreducible decomposition $C = \sum_j c_j C_j$. Then the pair is said to be *LC* (resp. *KLT*) if $c_j \leq 1$ (resp. $c_j < 1$) for all j . This definition does not depend on the choice of μ . The pair is called

DLT if there exists a log resolution μ such that $c_j \leq 1$ for all j with strict inequalities when $\text{codim } \mu(C_j) \geq 2$.

The cone theorem ([3]) can be stated in the following way:

Theorem 1. *Let $f : X \rightarrow S$ be a projective morphism from a DLT pair (X, B) . Then the cone of nef divisors $\text{Nef}(X/S)$ in $N^1(X/S)$ looks locally rational polyhedral when observed from the numerical class of $K_X + B$ in the following sense: let V be the part of the boundary $\partial \text{Nef}(X/S)$ which is visible from the point $v_0 = [K_X + B]$*

$$V = \{v \in \partial \text{Nef}(X/S) \mid [v, v_0] \cap \text{Nef}(X/S) = \{v\}\}$$

then any compact subset in the relative interior of V consists of finitely many faces which are defined by linear equations with rational coefficients.

Let (X, B) be an LC pair consisting of a normal \mathbf{Q} -factorial variety and an \mathbf{R} -divisor with a projective morphism $f : X \rightarrow S$ to a base space. We assume an additional condition that there exists another boundary \bar{B} such that (X, \bar{B}) is a KLT pair. Then the MMP or the MMP with scaling for the morphism $f : (X, B) \rightarrow S$ works and preserves the situation in the sense that the cone and contraction theorems hold and the resulting morphism after a divisorial contraction or a flip satisfies the same condition. The existence of flips is proved in [2], but the termination of flips in general is not yet.

A *minimal model* for a projective morphism $f : (X, B) \rightarrow S$ is defined to be a projective morphism $g : (Y, C) \rightarrow S$ with a birational map $\alpha : X \dashrightarrow Y$ over S which satisfies the following conditions:

1. α is surjective in codimension one, $C = \alpha_* B$ is the strict transform, and (Y, C) is a \mathbf{Q} -factorial LC pair.
2. $K_Y + C$ is nef over S .
3. Any exceptional prime divisor for α has reason to be contracted in the sense that it has positive coefficient in the difference of $K_X + B$ and $K_Y + C$. More precisely, if $p : V \rightarrow X$ and $q : V \rightarrow Y$ are common resolutions such that $q = \alpha \circ p$, then $p^*(K_X + B) - q^*(K_Y + C)$ has positive coefficient at the strict transform of an arbitrary prime divisor on X whose image on Y has higher codimension.

The condition 3 is referred to as the *negativity* of $K_X + B$ for α .

If a minimal model exists, then $K_X + B$ is pseudo-effective. The converse, the existence of a minimal model, is still a conjecture. The conjecture is proved to be true if the boundary B is big in [1].

A *canonical model* is defined to be a projective morphism $h : Z \rightarrow S$ with a surjective morphism $g : Y \rightarrow Z$ with connected geometric fibers from a minimal model such that $K_Y + C = g^*H$ for an \mathbf{R} -Cartier divisor H on Z which is ample over S .

Minimal models are not necessarily unique for a given morphism $f : (X, B) \rightarrow S$. But any minimal models are equivalent in the following sense: if $\alpha_i : (X, B) \dashrightarrow (Y_i, C_i)$ for $i = 1, 2$ are minimal models and $p_i : V \rightarrow Y_i$ are common resolutions such that $p_1^{-1} \circ \alpha_1 = p_2^{-1} \circ \alpha_2$, then we can prove that $p_1^*(K_{Y_1} + C_1) = p_2^*(K_{Y_2} + C_2)$ as a consequence of the Hodge index theorem. The negativity condition implies that the Y_i are isomorphic in codimension one. On the other hand, the canonical model is unique in the following sense: the composite rational map to the canonical model $\beta = g \circ \alpha$ is uniquely determined by the given morphism $f : (X, B) \rightarrow S$.

3 Numerical linear system

We propose definitions of numerical linear systems and numerical Zariski decompositions. They are easier to deal with than \mathbf{R} -linear systems as in [1] and sectional decompositions as in [4] and [7].

Let $f : X \rightarrow S$ be a projective morphism from a \mathbf{Q} -factorial normal variety, and D an \mathbb{R} -Cartier divisor which is pseudo-effective over S , i.e., $[D] \in \text{Psef}(X/S)$. We define the *numerically fixed part* of D by

$$N(D) = \lim_{\epsilon \downarrow 0} (\inf \{ D' \mid D' \equiv D + \epsilon A, D' \geq 0 \})$$

where A is an arbitrarily fixed ample divisor, $D' \geq 0$ means that the \mathbf{R} -divisor D' is effective, and the infimum of \mathbf{R} -divisors is defined by the infimums of coefficients. We note that $N(D)$ does not depend on the choice of A . We also note that there are only finitely many irreducible components of $N(D)$, since they are linearly independent in $N^1(X/S)$. For example, if D is nef, then $N(D) = 0$.

A similar statement to the following lemma for the case of the \mathbf{R} -linear systems is proved in [1] using a result in [7]. We believe that our approach is more natural and easier.

Lemma 2. *Let $f : (X, B) \rightarrow S$ be a projective morphism from a \mathbf{Q} -factorial LC pair, and let $\alpha : (X, B) \dashrightarrow (Y, C)$ be a minimal model. Then the prime divisors contracted by α are precisely the irreducible components of the numerically fixed part of $K_X + B$.*

Proof. Let $p : \tilde{X} \rightarrow X$ and $q : \tilde{X} \rightarrow Y$ be a common resolution. Then $p^*(K_X + B) - q^*(K_Y + C)$ is effective, and the coefficients of the exceptional divisors of α are positive.

Since $K_Y + C$ is nef, there is no numerically fixed part;

$$\inf\{D'_Y \mid D'_Y \equiv K_Y + C + \epsilon A_Y, D'_Y \geq 0\} = 0$$

for any $\epsilon > 0$ and any ample divisor A_Y on Y . Because of the negativity of $K_X + B$ for α , the difference $(K_X + B + \epsilon \alpha_*^{-1} A_Y) - p_* q^*(K_Y + C + \epsilon A_Y)$ is still effective if ϵ is small enough, where $\alpha_*^{-1} A_Y$ is the strict transform of A_Y . Then it follows that

$$\inf\{D' \mid D' \equiv K_X + B + \epsilon \alpha_*^{-1} A_Y, D' \geq 0\}$$

is exceptional for α , and so are any irreducible components of the numerically fixed part of $K_X + B$.

Conversely, if $D' \equiv K_X + B + \epsilon A$, then $\alpha_* D' \equiv K_Y + C + \epsilon \alpha_* A$. Hence $D' - p_* q^* \alpha_* D'$ has positive coefficients on any exceptional divisors of α if ϵ is small enough. Therefore any exceptional divisors of α are numerically fixed for $K_X + B$. \square

4 Cone decompositions

A *polytope* is a closed convex hull of finitely many points in a real vector space. It is called *rational* if these points have rational coordinates.

We start with a polytope decomposition with respect to the canonical models:

Theorem 3. *Let (X, \bar{B}) be a \mathbf{Q} -factorial KLT pair with a projective morphism $f : X \rightarrow S$ to a base space, B_1, \dots, B_r effective \mathbf{Q} -Cartier divisors, and \tilde{V} a polytope in the space $\{B = \sum_i b_i B_i \mid b_i \in \mathbf{R}\} \cong \mathbf{R}^r$ such that the pairs (X, B) are LC for all $B \in \tilde{V}$. We consider a closed convex subset*

$$V = \{B \in \tilde{V} \mid [K_X + B] \in \text{Psef}(X/S)\}.$$

Assume that for each $B \in V$, there exist a minimal model $\alpha : (X, B) \dashrightarrow (Y, C)$ and a canonical model $g : Y \rightarrow Z$ for $f : (X, B) \rightarrow S$. Moreover assume that there exists a real number $\epsilon > 0$ for each given $B \in V$ with $\alpha : X \dashrightarrow Y$ and $g : Y \rightarrow Z$ as above, such that the morphism $g : (Y, \alpha_* B') \rightarrow Z$ for $B' \in \tilde{V}$ has minimal and canonical models whenever $[K_Y + \alpha_* B'] \in \text{Psef}(Y/Z)$ and $\|B' - B\| \leq \epsilon$, where $\|\cdot\|$ denotes the maximum norm of the coefficients. Then there exists a finite decomposition to disjoint subsets

$$V = \coprod_{j=1}^s V_j$$

and rational maps $\beta_j : X \dashrightarrow Z_j$ which satisfies the following conditions:

1. $B \in V_j$ if and only if β_j gives the canonical model for $f : (X, B) \rightarrow S$.
2. The closures \bar{V}_j , hence V , are polytopes for all j . Moreover, if \tilde{V} is a rational polytope, then so are the \bar{V}_j and V .

Proof. We use an idea of Shokurov [8]. We proceed by induction on the dimension of V . If $\dim V = 0$, then the assertion is clear. Assume that $\dim V > 0$. We fix an arbitrary point $B_0 \in V$. Let $\alpha_0 : (X, B_0) \dashrightarrow (Y_0, C_0)$ be a minimal model and $g_0 : (Y_0, C_0) \rightarrow Z_0$ a canonical model. We can write $K_{Y_0} + C_0 = g_0^* H_0$ for an \mathbf{R} -Cartier divisor H_0 on Z_0 which is ample over S .

We take a real number $\epsilon > 0$ such that $g_0 : (Y_0, \alpha_{0*} B) \rightarrow Z_0$ has a minimal model $\alpha : (Y_0, \alpha_{0*} B) \dashrightarrow (Y, C)$ and a canonical model $g : (Y, C) \rightarrow Z$ with $h : Z \rightarrow Z_0$ if $K_{Y_0} + \alpha_{0*} B$ is pseudo-effective over Z_0 and $\|B - B_0\| \leq \epsilon$. If ϵ is sufficiently small, then $K_X + B$ is negative for α_0 .

We can write $K_Y + C = g^* H$ for an \mathbf{R} -Cartier divisor H on Z which is ample over Z_0 . If we take $\delta > 0$ sufficiently small, then $(1 - \delta)h^* H_0 + \delta H$ is ample over S . If we set $B' = (1 - \delta)B_0 + \delta B$, then $\alpha \circ \alpha_0 : (X, B') \dashrightarrow (Y, C')$ for $C' = (1 - \delta)\alpha_* C_0 + \delta C$ is a minimal model for $f : (X, B') \rightarrow S$ because the negativity still holds, and $g : (Y, C') \rightarrow Z$ is a canonical model because the ampleness holds.

We take a polytope \tilde{U} inside \tilde{V} which contains B_0 in the relative interior and which is contained in the above ϵ neighborhood. We take \tilde{U} to be rational when \tilde{V} is rational. Let

$$U = \{B \in \tilde{U} \mid [K_{Y_0} + \alpha_{0*} B] \in \text{Psef}(Y_0/Z_0)\}$$

Then $V \cap \tilde{U} \subset U$. By the induction assumption, the boundary ∂U is a union of polytopes, and there is a finite polytope decomposition of ∂U which corresponds to the classification of canonical models of $g_0 : (Y_0, \alpha_{0*}B) \rightarrow Z_0$. Moreover it is rational if so is \tilde{V} . It also follows that $U \subset V$.

If \tilde{U} was chosen sufficiently small, then the cones over these polytopes with the vertex B_0 give a finite polytope decomposition of U , which is rational if \tilde{V} is. Since V is compact, it is covered by finitely many such U 's, and the assertion is proved. \square

Next we consider a further polytope decomposition with respect to the minimal models:

Theorem 4. *Assume the conditions of the above theorem. Then each V_j has further decomposed to a finite disjoint union*

$$V_j = \coprod_{k=1}^t W_{j,k}$$

which satisfies the following conditions: let $\alpha : X \dashrightarrow Y$ be a birational map such that

$$W = \{B \in V \mid \alpha \text{ is a minimal model for } (X, B)\}$$

is non-empty. Then

1. *There exists an index j such that $W \subset \bar{V}_j$.*
2. *If $W \cap V_j$ is non-empty for some j , then $W \cap V_j$ coincides with one of the $W_{j,k}$.*
3. *The closure $\bar{W}_{j,k}$ is a polytope for any j and k . Moreover, if \tilde{V} is a rational polytope, then so are the $\bar{W}_{j,k}$.*

We note that there may be infinitely many W 's such that $W \cap V_j = W_{j,k}$ for fixed j, k .

Proof. 1. If $\alpha_*(K_X + B_i)$ is nef for $i = 1, 2$, then so is $\alpha_*(K_X + tB_1 + (1-t)B_2)$ for $t \in [0, 1]$. The negativity descends as well, hence W is a convex subset of V .

We take a point $B \in W$ in the relative interior, and let $g : Y \rightarrow Z$ be the canonical model for (Y, α_*B) . Then $g^*\text{Nef}(Z/S)$ is a face of $\text{Nef}(Y/S)$,

and $[\alpha_*(K_X + B)] \in g^* \text{Amp}(Z/S)$. Since $[\alpha_*(K_X + B')] \in \text{Nef}(Y/S)$ for all $B' \in W$, it follows that $[\alpha_*(K_X + B')] \in g^* \text{Amp}(Z/S)$ if B' is a point in the relative interior. Therefore $W \subset \bar{V}_j$ if V_j corresponds to $g \circ \alpha$.

2. Let $\alpha_i : X \dashrightarrow Y_i$ for $i = 1, 2$ be birational maps, and let W_i be the corresponding subsets of V which are assumed to be non-empty. Assume that there are morphisms $g_i : Y_i \rightarrow Z$ such that $\beta = g_1 \circ \alpha_1 = g_2 \circ \alpha_2$ corresponds to some V_j . Let $\gamma : Y_1 \dashrightarrow Y_2$ be the birational map such that $\alpha_2 = \gamma \circ \alpha_1$. We claim that, if γ is an isomorphism in codimension one, then $W_1 \cap V_j = W_2 \cap V_j$.

Indeed, if $B \in W_1 \cap V_j$, then $K_{Y_1} + \alpha_{1*}B = g_1^*H$ for some H on Z which is ample over S . Since γ is an isomorphism in codimension one, it follows that $K_{Y_2} + \alpha_{2*}B = g_2^*H$, hence $K_{Y_2} + \alpha_{2*}B$ is nef and equivalent to $K_{Y_1} + \alpha_{1*}B$. The negativity for the exceptional divisors holds at the same time for Y_1 and Y_2 . Therefore $B \in W_2 \cap V_j$.

Let $\{E_m\}$ be the set of all the prime divisors E_m each of which is contained in the numerically fixed part of $K_X + B'$ for B' being a vertex of \bar{V}_j . By Lemma 2, this is a finite set. Any numerically fixed component of $K_X + B$ for arbitrary $B \in V_j$ belongs to this set. Therefore there are only finitely many possibilities for the set of prime divisors which coincides with the set of numerically fixed components of $K_X + B$ for some $B \in V_j$. Hence the decomposition is finite.

3. The negativity condition of a prime divisor with respect to a birational map is expressed by a linear inequality with rational coefficients. Thus each chamber $W_{j,k}$ is bounded by linear equalities with rational coefficients of the following two types: (a) those coming from the negativity condition for the corresponding minimal model in the case where the boundary is open, and (b) those from other minimal models in the case of closed boundaries. Hence we have the assertion. \square

Corollary 5. *A sequence of flips in the MMP with scaling terminates if minimal and canonical models exist along the line segment in the space of divisors corresponding to this MMP process.*

Proof. The chambers V_j are finite in number. Therefore we may fix one V_j in order to prove the termination. Let $\alpha : X^- \dashrightarrow X^+$ be a flip in the MMP for a pair (X, B) with scaling H , and let t be the real number such that $K_{X^-} + B + tH$ is numerically trivial for the flip.

Suppose that both $(X, B + (t + \epsilon)H)$ and $(X, B + (t - \epsilon)H)$ belong to the same chamber V_j . Thus X^- and X^+ are minimal models respectively

for these pairs. Let $g^\pm : X^\pm \rightarrow Z$ be the canonical models. There are ample \mathbf{R} -divisors H^\pm on Z such that $K_{X^\pm} + B + (t \mp \epsilon)H = (g^\pm)^*H^\pm$. Since α is an isomorphism in codimension one, it follows that $K_{X^\mp} + B + (t \mp \epsilon)H = (g^\mp)^*H^\pm$. In other words, $K_{X^\pm} + B + (t \pm \epsilon)H$ is nef. But this is a contradiction. Therefore there does not exist a flip inside the same chamber V_j . \square

Corollary 6. *Let $f : (X, B) \rightarrow S$ be a projective morphism from a KLT pair, and let $\alpha_i : (X, B) \dashrightarrow (Y_i, C_i)$ ($i = 1, 2$) be two minimal models. Assume that there exists their canonical model. Then they are connected by a sequence of flops.*

We note that the boundary B may not be a \mathbf{Q} -divisor.

Proof. The Y_i are isomorphic in codimension one by Lemma 2. Let $g_i : Y_i \rightarrow Z$ be the morphisms to the canonical model. Let H_i be ample \mathbf{R} -divisors on the Y_i . Let V be the triangle spanned by \mathbf{R} -divisors 0 , H_1 and H_2 on Y_1 , where we use the same symbol for the strict transforms as usual. We may assume that the (Y_i, H_i) are KLT.

There are finitely many subsets V_j of V corresponding to the canonical models by the theorem. If we replace the H_i by ϵH_i for sufficiently small ϵ , then we may assume that all the \bar{V}_j contain 0 . Then the MMP with scaling corresponding to the line segment joining $K_{Y_1} + H_1$ and $K_{Y_1} + H_2$ gives the desired sequence of flops. \square

We close this paper with some examples in order to illustrate the theorems.

Example 7. Let $f_0 : X_0 \rightarrow S$ be a ruled surface with two disjoint sections S_0 and S_1 over a curve, $\alpha_0 : X \rightarrow X_0$ a blowing up at a point on S_0 , and $f : X \rightarrow S$ the composite map. There is one singular fiber of f which will be denoted by $C_0 + C_1$, where C_0 is the exceptional curve for α_0 .

Let $V = \{B = \frac{1}{2}C_0 + tC_1 \mid 0 \leq t \leq 1\}$. Then the canonical models for the (X, B) are reduced to the identity to S for all $B \in V$. Let W_- , W_0 and W_+ be the subsets of V defined by equations $t < \frac{1}{2}$, $t = \frac{1}{2}$ and $t > \frac{1}{2}$, respectively. They correspond to minimal models $\alpha_0 : X \rightarrow X_0$, $X \rightarrow X$, and $\alpha_1 : X \rightarrow X_1$, the contraction morphism of C_1 .

Example 8. Let X^- be a Calabi-Yau threefold and let $\alpha : X^- \rightarrow Y \leftarrow X^+$ be a flop such that there is a non-canonical isomorphism $h : X^- \cong X^+$ as in

Example 10. Let H^- be an ample divisor on X^- , and let $H^+ = \alpha_* H^-$ be the strict transform which is also considered to be a divisor on X^- by using h . If $\epsilon > 0$ is sufficiently small, then (X^-, B_t) for $B_t = \epsilon((1-t)H^- + tH^+)$ with $t \in [0, 1]$ are KLT. We have a decomposition $V = V^- \coprod V^0 \coprod V^+$, where $V = [0, 1]$, $V^- = [0, 1/2)$, $V^0 = \{1/2\}$ and $V^+ = (1/2, 1]$, corresponding to the canonical models given by X^- , Y and X^+ . The chambers $W^- = [0, 1/2]$ and $W^+ = [1/2, 1]$ correspond to minimal models X^- and X^+ . When $t = 1/2$, the canonical model Y has two minimal models X^\pm .

We recall two examples from [5] Example 3.8 in which there are infinitely many cones in the space of numerical equivalence classes of \mathbf{R} -divisors. In this sense we can say that our theorems deal with only “local” situations.

Example 9. Let $f : X \rightarrow S$ be the versal deformation of a singular fiber of type I_2 on an elliptic surface, i.e., a curve C which has two irreducible components isomorphic to \mathbf{P}^1 intersecting transversally at two points. We have $\dim X = 3$ and $\dim N^1(X/S) = 2$.

There are coordinates on $N^1(X/S)$ defined by $x = (D \cdot C_1)$ and $y = (D \cdot C_2)$ where the C_i are irreducible components of C . The pseudo-effective cone is the half space defined by $x + y \geq 0$. The effective cone is not closed because points $(1, -1)$ and $(-1, 1)$ are not represented by any effective \mathbf{R} -divisors.

The rays generated by $(n+1, -n)$ for $n \in \mathbf{Z}$ divide the pseudo-effective cone into infinitely many polytopes which correspond to canonical models.

Example 10. Let X be a generic hypersurface of degrees $(2, 2, 3)$ in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$. The pseudo-effective cone is the cone over the convex hull of the points $A = (0, 0, 1)$ and $C_n = (n+1, -n, \frac{3}{2}n(n+1))$ for $n \in \mathbf{Z}$, because its boundary points correspond to fibrations to lower dimensional spaces, i.e., elliptic fibrations and $K3$ fibrations. The effective cone is closed because the points A and C_n are represented by effective divisors. But it is divided into infinitely many polytopes again. The projection $X \rightarrow S = \mathbf{P}^1 \times \mathbf{P}^1$ induces a linear map $N^1(X) \rightarrow N^1(X/S)$ given by $(x, y, z) \mapsto (x, y)$, and the effective cone in the previous example is the image under this map.

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